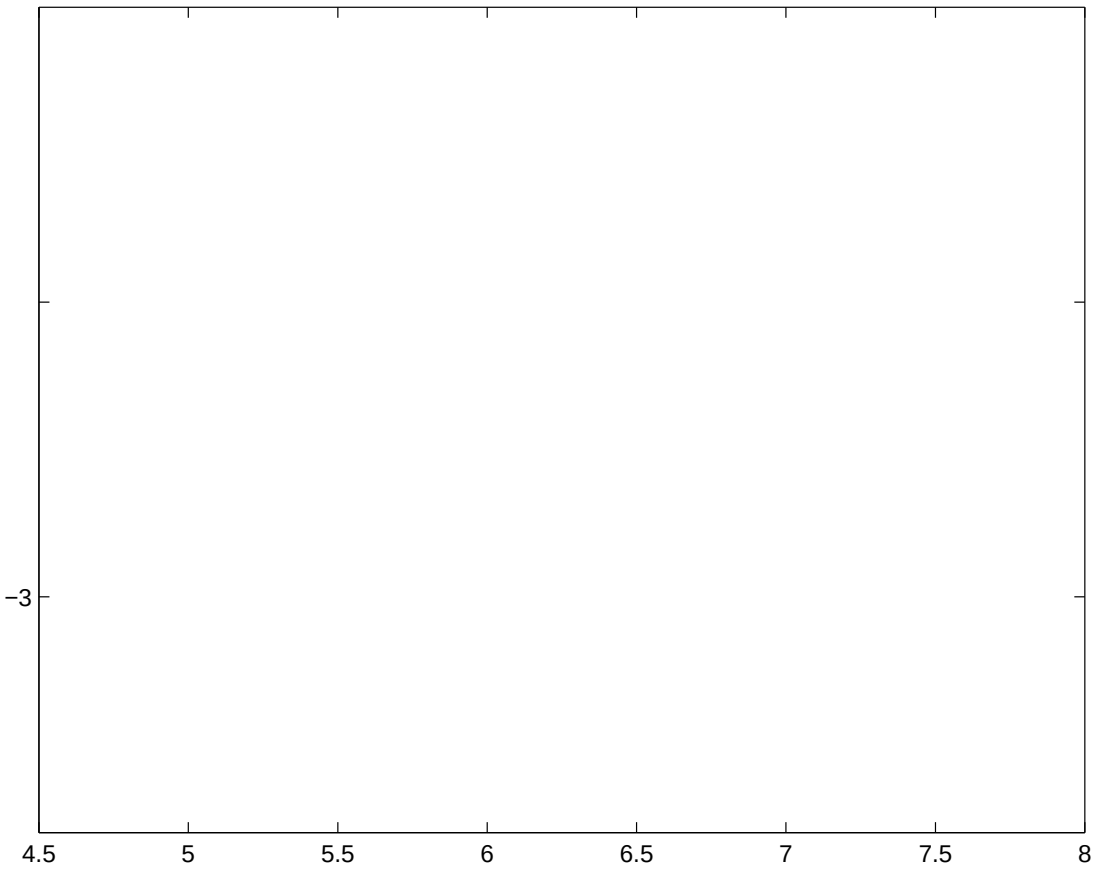
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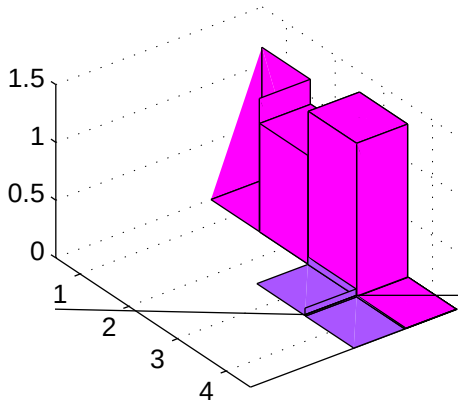
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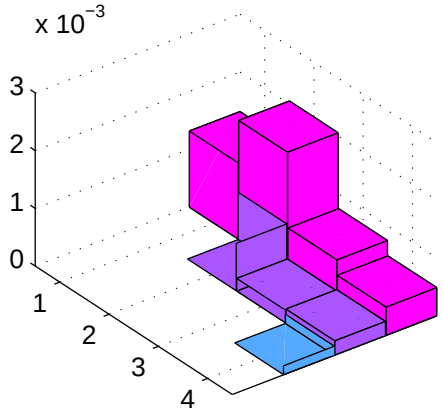
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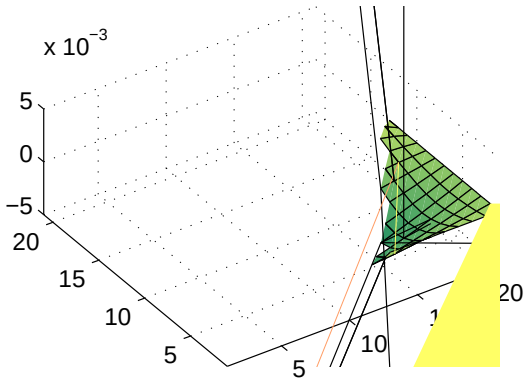
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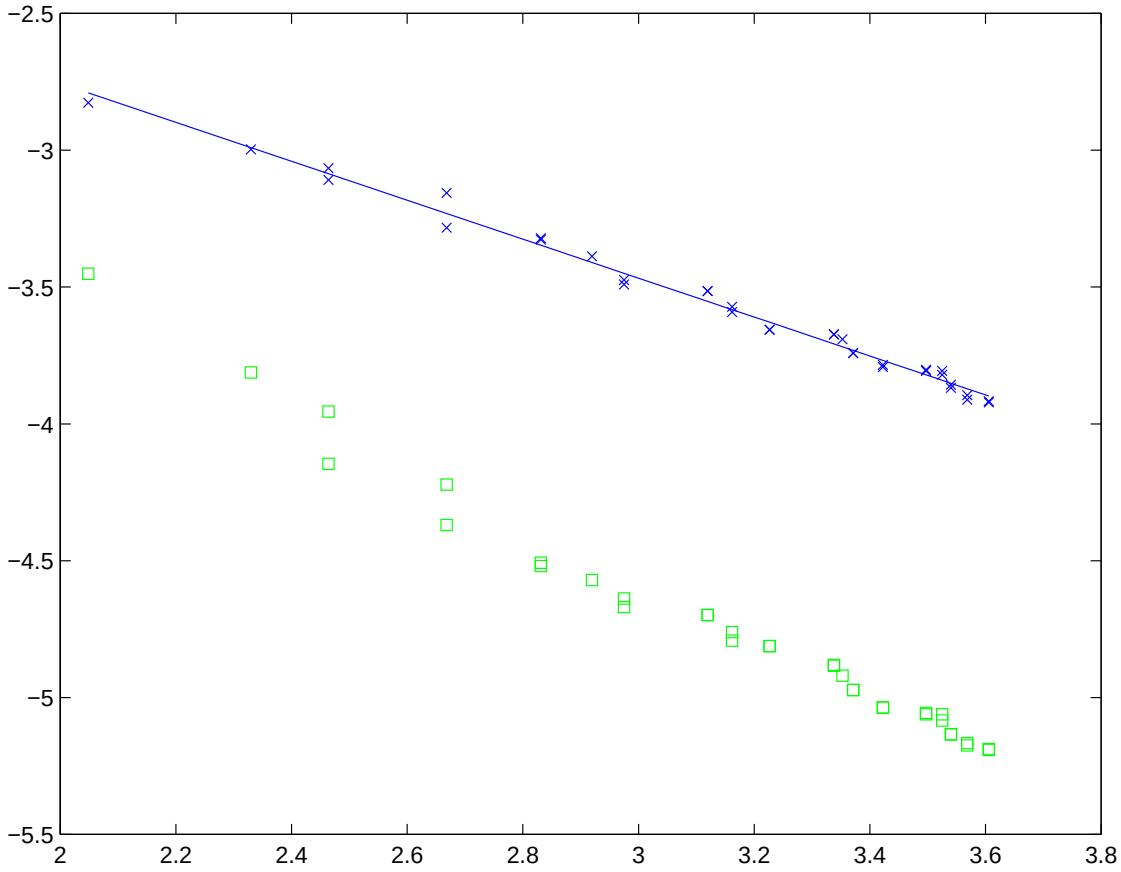
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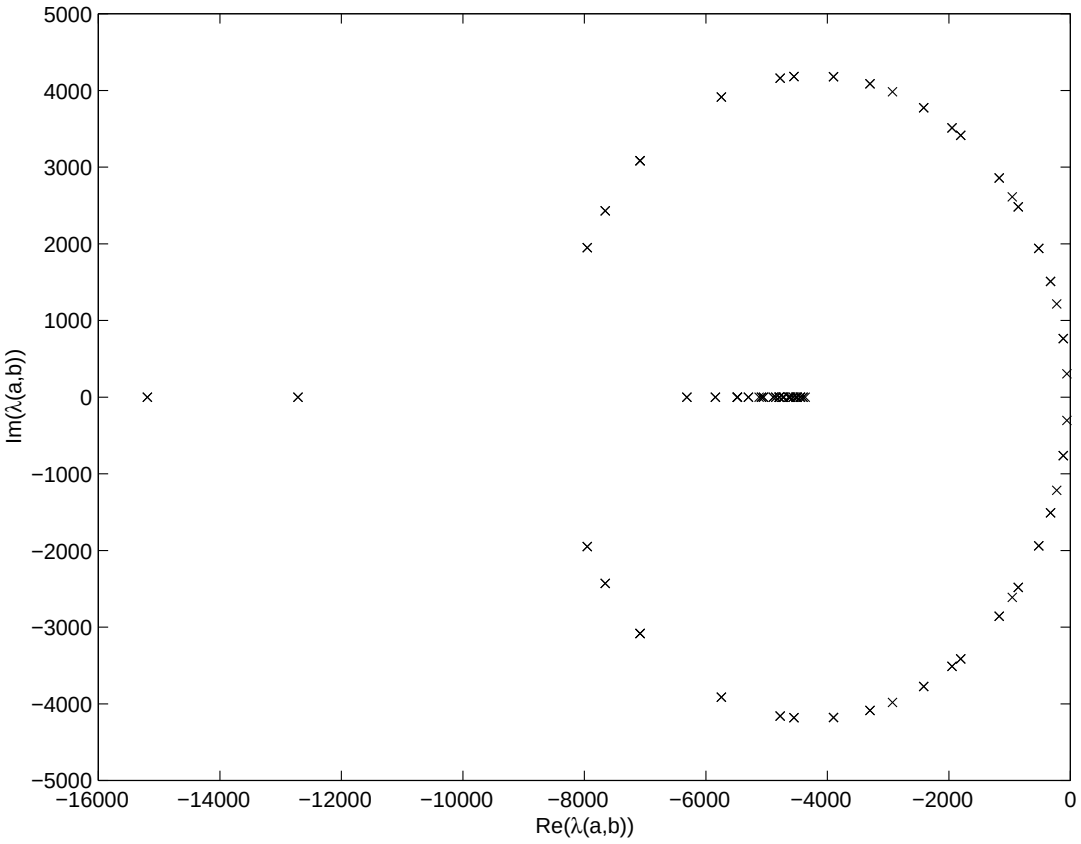
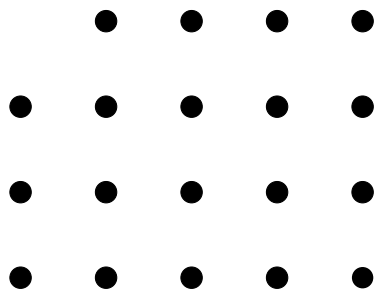
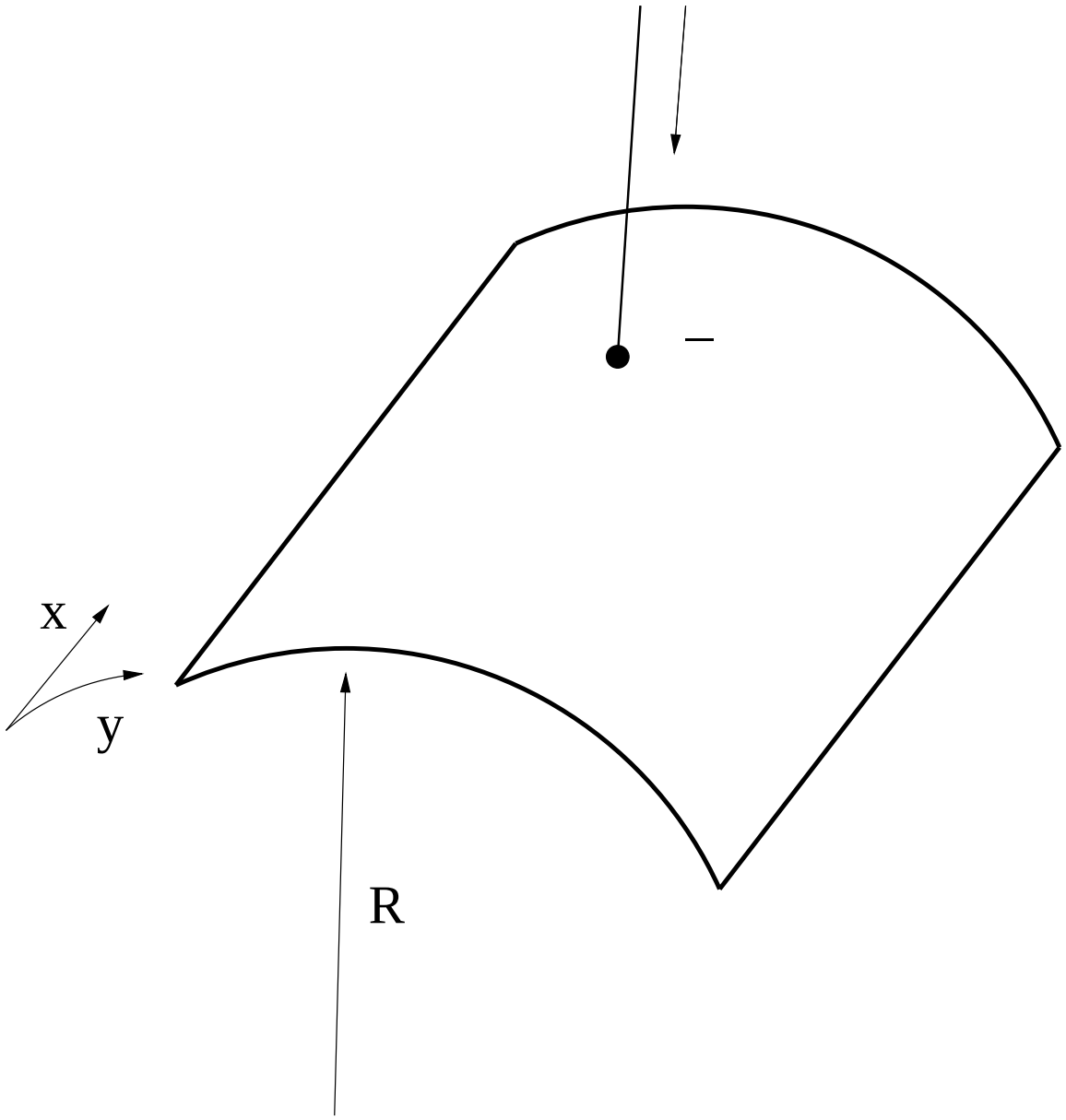


Figure 3: Partial set of eigenvalues (19) of the linearisation of the von Karman equations (7)-(8) about the rest state for $n = 9$; other parameters are as in Section 5. There are more real eigenvalues that are more negative than those shown.

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8 Conclusion

We have demonstrated an application of the ideas of post–processed Galerkin methods to the large set c ordinary differential equations resulting froma semi–discrete finite difference approximation to the von Karman equations that govern the vibrations c thin shells. We have shown that post–processing gives results of similar accuracy to those obtained from a nonlinear Galerkin method yet only requires a similar amount c computational power to that used for a flat Galerkin of lower order and is thus more efficient than both the flat Galerkin and nonlinear Galerkin methods. We believe that post–processing c this formis a useful technique and will be c use in the many areas for which the numerical solution c partial differential equations is necessary.

Acknowledgements

We would like to thank Professor Edriss Titi for telling us about post–processed Galerkin methods, and the referees for their useful comments. This work was supported by the EPSRC Applied Nonlinear Mathematics Initiative.

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functional form of (22) and (23) — for the von Karman equations they are both cubic polynomials in the components of y_k and linear in $f(t)$. For our first method, let us set $k = 1$, assume that $M(1) = 2$ and that we are trying to find (22). Write

$$y_1 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

We know that the first component of (22) can be written

$$\dot{u}_1 = \sum_{i=1}^{2} a_i u_i + \sum_{i=1}^{2}\sum_{j=i}^{2} b_{i,j} u_i u_j + \sum_{i=1}^{2}\sum_{j=i}^{2}\sum_{k=j}^{2} c_{i,j,k} u_i u_j u_k + A\sin(\omega t) \qquad (25)$$

There are $P \equiv [M(k)]^3/6 + [M(k)]^2 + 11M(k)/6$ unknown coefficients $a_1, \ldots, c_{2,2,2}$ in (25), so if we choose P random $M(k)$–tuples $[u_1(s) \cdots u_{M(k)}(s)]$ for $s = 1, \ldots, P$ and create a matrix, H (for $P = 9$ in this case) which can be thought of as a generalisation of a Vandermonde matrix [19]

$$\begin{pmatrix} u_1(1) & u_2(1) & u_1^2(1) & u_1(1)u_2(1) & u_2^2(1) & u_1^3(1) & u_1^2(1)u_2(1) & u_1(1)u_2^2(1) & u_2^3(1) \\ u_1(2) & u_2(2) & u_1^2(2) & u_1(2)u_2(2) & u_2^2(2) & u_1^3(2) & u_1^2(2)u_2(2) & u_1(2)u_2^2(2) & u_2^3(2) \\ \vdots & & & & \vdots & & & & \vdots \\ u_1(P) & u_2(P) & u_1^2(P) & u_1(P)u_2(P) & u_2^2(P) & u_1^3(P) & u_1^2(P)u_2(P) & u_1(P)u_2^2(P) & u_2^3(P) \end{pmatrix}$$

and a vector

$$X = \begin{pmatrix} \dot{u}_1([u(1);0;0],0) \\ \dot{u}_1([u(2);0;0],0) \\ \vdots \\ \dot{u}_1([u(P);0;0],0) \end{pmatrix}$$

where $\dot{u}_1$ is the first component of (22

be thought of as showing the “

We make an analogy with the PDE by assuming that there is a graph relating the asymptotic behaviour of the higher modes ($y_{k,m}$ for $k < m \leq \eta$) to the behaviour of the lower modes (y_k), i.e. there exists a $\Psi_{k,m}$ such that

$$y_{k,m} = \Psi_{k,m}(y_k)$$

Again, in analogy with the construction for the PDE, we take a first approximation to $\Psi_{k,m}$, $\Psi^1_{k,m}$ defined as

$$y_{k,m} = \Psi^1_{k,m}(y_k) \equiv -(\Upsilon_{k,m})^{-1}\Gamma_{k,m}([y_k; 0; 0], t) \tag{23}$$

Thus the evolution on the approximate inertial manifold is given by the system

$$\dot{y}_k = (\Upsilon_k)y_k + \Gamma_k([y_k; -(\Upsilon_{k,m})^{-1}\Gamma_{k,m}([y_k; 0; 0], t); 0], t) \tag{24}$$

with the solution reconstructed as $v \approx y_k + \Psi^1_{k,m}(y_k)$. The post–processing approach consists of solving (22) and then reconstructing the solution as $v \approx y_k + \Psi^1_{k,m}(y_k)$. Note that since $\Upsilon_{k,m}$ is block diagonal, it is trivial to invert.

Two advantages of using a finite–difference approach as opposed to a spectral method are that for finite–difference, the domain need not be regular and also that the eigenfunctions of the differential operator L need not be found analytically. The "eigenfunctions" are instead the eigenvectors of B (the z_js) which are determined numerically from the semi–discrete system (16).

Note that the construction above is very similar to the construction of a centre manifold for an ODE — see, e.g. [21]. Also, the idea of constructing an inertial manifold for a large set of ODEs derived from a finite difference scheme is not new and was discussed in, e.g. [6].

5 Numerical results

In this section we show some numerical results regarding the finite difference model discussed above. In Figure 4 we compare the accuracy of the three methods — flat Galerkin (FG), post–processed Galerkin (PPG) and the nonlinear Galerkin method (NLG). We have chosen $n = 9$ in this example, with forcing at node (2,1) with $\mu f(t) = 10000 \sin(800t)$. Other constants have the values

μ	β	β_2	ν	E	h	R
78.5 kg/m^2	7071 Ns/m	$\beta/(16\pi^4)$	0.3	2.1×10^{11} N/m^2	0.01 m	8.333 m

for $j = 1, \ldots, \eta$, where $m(j)$ is the dimensionality of the eigenspace corresponding to the eigenvalues represented by C_j. With this definition, the dynamics obtained by restricting (20) to the first j modes is giv

For any reasonable value of n, (16) is a high–dimensional system (of dimension $2n^2$) which is impractical to study in any depth. This is the motivation behind trying to use some of the ideas relating to approximate inertial manifolds that have been developed for infinite–dimensional PDEs to help us in our study of (16).

We are trying to make an analogy between (16) and a generic PDE (1). Instead of using the eigenfunctions of L as a basis for the space of solutions, we use the numerically determined eigenvectors of the linearisation of (16) about the rest state (the origin). To do this we write (16) as

$$\dot{u} = Bu + H(u,t)$$

where B is the spatial Jacobian of $G(u,t)$ evaluated at $u = 0$ and $H(u,t) = G(u,t) - Bu$. For large enough n, the eigenvalues of B come in a mixture of complex conjugate pairs and simple reals which we order according to their real parts, for example

$$\{-\rho_1 \pm i\omega_1, -\rho_2, \ldots, -\rho_k \pm i\omega_k, \ldots, -\rho_m, \ldots, -\rho_\eta\} \tag{17}$$

where $0 < \rho_1 \leq \rho_2 \leq \ldots \leq \rho_\eta$ and $\rho_1 < \rho_\eta$, i.e. not all the ρ_js are equal, and $n^2 \leq \eta < 2\hat{n}$

now vectors representing their discretisations over all grid points. The application of (8)-(11) provides only $n^2 + 8n + 12$ equations, so we use a second approximation to the boundary condition

$$\frac{\partial^2 \phi}{\partial x \partial y} = 0$$

at all 4 corners, *viz.*

$$[\phi_{i,j} + \phi_{i+1,j+1} -$$

The boundary conditions corresponding to simply–supported edges are

$$\frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{at} \quad x = 0, a \tag{9}$$

$$\frac{\partial^2 \phi}{\partial x^2} = 0 \quad \text{at} \quad y = 0, a \tag{10}$$

$$\frac{\partial^2 \phi}{\partial x \partial y} = 0 \quad \text{at} \quad y = 0, a; x = 0, a \tag{11}$$

$$\frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at} \quad x = 0, a \tag{12}$$

$$\frac{\partial^2 w}{\partial y^2} = 0 \quad \text{at} \quad y = 0, a \tag{13}$$

$$w = 0 \quad \text{at} \quad y = 0, a; x = 0, a \tag{14}$$

where a is the side length of the panel. (We set a =1m and do not refer to it again.)

In order to implement a finite–difference scheme we lay a regular cartesian grid over the domain, as shown in Figure 2, and approximate the spatial derivatives using standard central difference formulae:

$$\begin{aligned}
\nabla^4 \psi_{i,j} &\approx [20\psi_{i,j} - 8(\psi_{i+1,j} + \psi_{i-1,j} + \psi_{i,j-1} + \psi_{i,j+1}) \\
&+ 2(\psi_{i-1,j-1} + \psi_{i-1,j+1} + \psi_{i+1,j-1} + \psi_{i+1,j+1}) \\
&+ \psi_{i-2,j} + \psi_{i+2,j} + \psi_{i,j-2} + \psi_{i,j+2}]/(\delta x)^4 \\
\left.\frac{\partial^2 \psi}{\partial y^2}\right|_{i,j} &\approx [\psi_{i,j-1} - 2\psi_{i,j} + \psi_{i,j+1}]/(\delta x)^2 \\
\left.\frac{\partial^2 \psi}{\partial x^2}\right|_{i,j} &\approx [\psi_{i-1,j} - 2\psi_{i,j} + \psi_{i+1,j}]/(\delta x)^2 \\
\left.\frac{\partial^2 \psi}{\partial x \partial y}\right|_{i,j} &\approx [\psi_{i+1,j+1} + \psi_{i-1,j-1} - \psi_{i+1,j-1} - \psi_{i-1,j+1}]/(2\delta x)^2
\end{aligned}$$

where δx is the grid spacing. We define the integer n by saying that there are n^2

ff A RR

8). The model we study is shown schematically in Figure 1. It is part of a cylindrical panel with radius R, forced at the point $(\bar{x}, \bar{y})$ with the function $f(t)$.

We should mention the paper by Foale et al. [3], which contains results similar to those presented here. Foale et al. studied a finite–difference model of a panel similar to that shown in Figure 1, the main difference being that their panel was axially forced on the boundaries rather than at a point. This may seem a minor difference, but it is thought that the efficiency of the nonlinear Galerkin method as opposed to the flat Galerkin method depends on (among other things) the smoothness of solutions of the PDE, which in turn depends on the smoothness of forcing [10]. Foale et al. [3] concluded that a nonlinear Galerkin method provided little advantage over a flat Galerkin method, but we feel that this was due to the very smooth solutions

In practice, Q_k is the projection onto an infinite–dimensional space, so a truncation of it must be used. Thus we define

$$\Psi^1_{k,m}(p) \equiv L^{-1}\{Q_{k,m}f - Q_{k,m}[R(p)]\}$$

where $Q_{k,m} \equiv I - P_k - P_m$ and $m > k$. Equation (4) now becomes

$$\frac{dv_k}{dt} + Lv_k + P_k[R(v_k + L^{-1}\{Q_{k,m}f - Q_{k,m}[R(v_k)]\})] = P_k f \tag{5}$$

and we reconstruct the solution as $u \approx v_k + \Psi^1_{k,m}(v_k)$.

We can think of (5) as an equation for the dynamics on the approximate inertial manifold. The last term on the left side of (5) normally requires a lot of effort to evaluate, and the question of the computational efficiency of integrating (5) as opposed to a Galerkin projection onto s modes, where $s > k$, to obtain a desired degree of accuracy is raised.

The work of Garcia-Archilla et al. [7, 8] on post–processing Galerkin methods shows that it is sometimes possible to obtain the accuracy of a system like (5) with no more effort than that involved in integrating a k–mode Galerkin system. The idea is to integrate a k–mode Galerkin approximation:

$$\frac{dy_k}{dt} + Ly_k + P_k[R(y_k)] = P_k f \tag{6}$$

and then when output is required, say at time T, "lift" this data up to the approximate inertial manifold, i.e. write $u(T) \approx y_k(T) + \Psi^1_{k,m}(y_k(T))$. Garcia-Archilla et al. [7, 8] showed that this is often as accurate as the solution $v_k(T)+\Psi^1_{k,m}(v_k(T))$ obtained from integrating (5), but has the advantage that the system (6) is simpler to integrate. The lifting of the solution onto the approximate inertial manifold need only be carried out when output is required, rather than at every time–step in the numerical integration, as is the case when integrating (5).

We have reviewed the theory of only spectral nonlinear Galerkin methods in this section — similar ideas have also been applied to both finite element [14, 15] and finite difference [20] schemes, although as far as we are aware, the idea of post–processing has not been applied to a finite element scheme, and this is the first application to a finite difference scheme.

4 Finite difference model of a panel

In this section we discuss one approach that combines the ideas of approximate inertial manifolds and finite–difference numerical schemes for the von Karman equations (7-

Defining $p \equiv P_k u$ and $q \equiv Q_k u$ and applying first P_k and then Q_k to (1) we obtain two coupled ODEs, the first finite–dimensional and the second infinite–dimensional:

$$\frac{dp}{dt} + Lp + P_k[R(p+q)] = P_k f \tag{2}$$

$$\frac{dq}{dt} + Lq + Q_k[R(p+q)] = Q_k f \tag{3}$$

The traditional Galerkin method corresponds to setting $q = 0$ in (2) and integrating the resulting finite–dimensional system:

$$\frac{dy_k}{dt} + Ly_k + P_k[R(y_k)] = P_k f$$

We then approximate the true solution of (1) by $u \approx y_k$.

The rationale behind the development of inertial manifolds is that if the attractor of (1) is finite–dimensional, then it is reasonable to hypothesise that for some k there exists a graph, Φ_k, such that on the attractor $q = \Phi_k(p)$, i.e. on the attractor the behaviour of all the higher modes is completely governed by the behaviour of a finite number of the lower modes. If this is the case, then to determine the dynamics on the attractor we substitute Φ_k into (2) and integrate the

Conceptually, nonlinear Galerkin methods split the infinite–dimensional phase space of the PDE into two complementary subspaces: a finite–dimensional one spanned by "slowly" contracting modes, and its infinite–dimensional complement. The dynamics in this infinite–dimensional space are then assumed to be "slaved" to the dynamics in the finite–dimensional space via an inertial manifold. Many computation schemes have been introduced where this slaving is used in the calculation of the dynamics. Recently, work by Garcia-Archilla et al. [7, 8] has shown that it is often more efficient to ignore the slaving when calculating the dynamics and only to use it when output from the system is actually required — this technique has been called "post–processing".

Much of the work relating to nonlinear Galerkin methods has been presented using spectral techniques. In this paper we make analogies between a large system of ordinary differential equations derived from a semi–explicit finite–difference scheme for the dynamics of a cylindrical panel forced at a point and a general PDE for which nonlinear Galerkin and post–processing techniques have been developed. In contrast with spectral methods, the finite–difference method we use can be applied to irregular domains. Our results regarding convergence rates and efficiency are similar to those obtained by other workers [7, 10] who have studied simpler PDEs using spectral methods.

3 Spectral nonlinear Galerkin methods

In this section we review the theory of spectral nonlinear Galerkin and post–processing methods with which we will later make analogies. For more details see [2, 4, 5, 7, 8, 9, 10].

Let us write our PDE as

$$\frac{du}{dt} + Lu + R(u) = f \tag{1}$$

where L is a linear spatial–differential operator, $R(u)$ contains nonlinear terms, and f is a forcing function. Assume that L has an infinite number of orthonormal eigenfunctions $\{w_i\}$

1 General Introduction

It is known that the solutions of some dissipative nonlinear partial differential equations (PDEs) evolve to a compact set known as a global attractor. Various schemes have been used for constructing finite systems of ordinary differential equations (ODEs) that reproduce the asymptotic dynamics of the original infinite dimensional PDE — one of the most well known being the Galerkin method. This method effectively ignores the small spatial structure of a solution less than a certain size, concentrating instead on the large scale structures.

During the past decade nonlinear Galerkin methods have been introduced which attempt to use the finite dimensionality of the attractor of the PDE to incorporate the influence of the small scale structures on the temporal evolution of the large scale structures. This method has often been successful in producing a greater level of accuracy when compared with a Galerkin method with the same spatial threshold, but often at an increased computational cost, and when comparing accuracy achieved for a given amount of computer time, it is not always clear that the nonlinear Galerkin method has any advantage.

This dilemma has recently been overcome by the introduction of the post–processed Galerkin method, where the influence of the small scale structures on the temporal evolution of the large scale structures is ignored, leading to a computational cost similar to that of an ordinary Galerkin method, until a computation is completed. At this stage the computed solution is "post–processed" (at small computational cost) to recover the small scale structure. This method is often (although not always) more efficient than either the traditional Galerkin or nonlinear Galerkin methods.

We make analogies between a PDE to which these three methods can be applied and a large set of ODEs derived from a finite difference scheme for computing the vibrations of a nonlinear shell and come to a similar conclusion regarding the efficiency of the post–processed Galerkin method.

2 Introduction

There has recently been much interest in the existence of inertial manifolds in classes of dissipative nonlinear partial differential equations (PDEs). Nonlinear Galerkin methods [13], which attempt to completely describe the dynamics on the attractor of a PDE with a *finite–dimensional* dynamical system, have arisen from this theory.

Abstract

We present the results of a computational study of the post–processed Galerkin methods put forward by Garcia-Archilla et al. [7, 8] applied to the nonlinear von Karman equations governing the dynamic response of a thin cylindrical panel periodically forced by a transverse point load.

We spatially discretise the shell using finite differences to produce a large system of ordinary differential equations. By analogy with spectral nonlinear Galerkin methods we split this large system into a "slowly" contracting subsystem and a "quickly" contracting subsystem. We then compare the accuracy and efficiency of (i) ignoring the dynamics of the "quick" system (analogous to a traditional spectral Galerkin truncation and sometimes referred to as "subspace dynamics" in the finite element community when applied to numerical eigenvectors), (ii) slaving the dynamics of the quick system to the slow system