Periodically-forced finite networks of heterogeneous globally-coupled oscillators: a low-dimensional approach

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[a]

particular network will synchronise [3,25]. Periodic forcing of systems is also ubiquitous [16,39], and so it is natural to study the entrainment of a network of coupled oscillators by a periodic forcing function. Many authors have studied small networks of two or three non-identical oscillators [3], and larger networks of oscillators that may have some symmetry [18,25] or a particular form of coupling [34]. The continuum limit in which there exists an infinite number of oscillators has also been studied in detail and many results are known for this case [2,4,17,24,38]. However, it is known that finite networks can show behaviour that does not occur in the continuum limit [4,12]. In many situations, finite networks are the most realistic way to model a physical system [6,12,35]. Results for large, finite networks will thus help bridge the gap between small network dynamics (for which bifurcation analysis is straightforward) and those for an infinite number of oscillators (where statistical physics provides the appropriate tools).

In this paper we consider a large but finite heterogeneous network of globally-coupled oscillators, which are collectively periodically forced. However, we do not analyse the system exactly; instead we analyse a low-dimensional description of it. This is the "equation-free" approach developed by Kevrekdis et al. [23]. The results here extend those of Moon et al. [30,32], obviating a reduction to phase oscillators: we consider two-variable (rather than single variable) oscillators, capable of undergoing Hopf bifurcations; we consider periodic forcing of the network, and we perform bifurcation analysis on the low-dimensional description of the system to understand how the behaviour of the full system changes as parameters are varied. We will not take the continuum limit as the number of oscillators tends to infinity, but instead analyse the realistic case of a finite number of oscillators.

The system we study is

$$\frac{dx_i}{dt} = y_i - x_i\left[x_i^2/3 - (\varphi + \beta\mu_i)\right] + x_i^2/2 - \frac{\square}{N}\sum_{j=1}^{N}(x_i - x_j) \qquad (1)$$

$$\frac{dy_i}{dt} = -x_i + A\sin(\omega t) \qquad (2)$$

for $i = 1, \ldots, N$, where N is the number of oscillators in the network. For most of this paper we set $N = 500$. The oscillators are van der Pol oscillators [19] with an extra term $(x_i^2/2)$ which breaks the internal symmetry $[(x, y) \to (-x, -y)]$ of eae586436(m)2.9246(m)2.9R.72(:)-169.6(s)4 rho

The μ_i are taken from a normal distribution with mean 0 and standard deviation 1. (As discussed below, the methodology can be used with other distributions.) If $\beta \neq 0$ the network is heterogeneous, and each oscillator, if uncoupled, would have a different angular frequency determined by the value of $\varphi + \beta\mu_i$. When $A = 0$, for β small enough, φ of moderate size and Γ large enough, the oscillators synchronise in the sense of having the same period. Note that oscillators i and j cannot synchronise in the sense of $x_i(t) = x_j(t)$ for all t unless $\mu_i = \mu_j$. In this synchronised state the attractor of the system is a periodic orbit, which could be parametrised by a periodic variable, say $\theta(t)$. The variables $x_1, \ldots x_N, y_1, \ldots, y_N$ could each then be written as functions of θ. This description would no longer be valid if one or more of the oscillators "unlocked" from the group.

We want to study the system in this synchronised state, but do not want to keep track of all the 2N variables $x_1, \ldots x_N, y_1, \ldots, y_N$. Instead, we describe the state of the system by a small number of variables. We cannot easily derive an equation that governs the dynamics of these variables, but by repeatedly mapping between the two levels of description of the system we can numerically evaluate the results of integrating these unavailable equations; we can also find their collectively periodic states and their dependence on parameters, without ever obtaining the reduced equations in closed form. This low-dimensional description results in computational savings by, for example, giving a much smaller Jacobian matrix.

If the system is in this synchronised state and we increase A from zero, it will become periodically driven and it may be possible for the oscillators to lock with the driving frequency [16]. The latter part of this paper will consider this phenomenon in detail, but we first discuss the particular lon inss

nomials of μ

Armed with these two operators we can now proceed to numerically solve the unavailable equation for the polynomial chaos coe□cients.

3 Coarse Projective Integration

Coarse projective integration entails accelerating the simulation of a high-dimensional system by projecting forward in time using only the variables in a low-dimensional description of the system. This is accomplished by occasionally performing short bursts of full simulation of the high-dimensional system in order to obtain the numerical information (estimation of the time-derivatives of the low dimensional description variables) required to perform accurate projections [23,30,32]. We can use the low-dimensional description in the previous section for coarse projective integration as follows. For convenience, let the high-dimensional description be the variable

$$X = [x_1, \ldots, x_N, y_1, \ldots, y_N] \in R^{2N} \tag{9}$$

and the low-dimensional polynomial chaos coe□cient description be the variable

$$Z = [a_0, \ldots, a_q, b_0, \ldots, b_q] \in R^{2(q+1)}. \tag{10}$$

Given $X(0)$, integrate (1)-(2) forward for N_1 steps of size δt. Calculate Z at some or all of the times $t = 0, \delta t, 2\delta t, \ldots, N_1\delta t$ using the restriction operator. Use these values of Z to extrapolate the values of Z to a time $N_2\delta t$ in the future, i.e. to time $(N_1 + N_2)\delta t$. Lift from the value of $Z((N_1 + N_2)\delta t)$ to $X((N_1 + N_2)\delta t)$ as detailed above. Restart the integration of (1)-(2) using $X((N_1 + N_2)\delta t)$ as the initial condition and integrate for a further N_1 time steps. Restrict to Z and repeat the procedure. If the cost of restricting, extrapolating and lifting is small compared to the cost of integrating the system (1)-(2) for N_2 time steps, this procedure may well be faster than integrating (1)-(2) directly. We 2632(r)-0.648378(.)-20rocedu. s9 [(s)4.79391(y)453TJ R187.97011Tf

are $(Z^j_{-3}, Z^j_{-2}, Z^j_{-1}, Z^j_0)$, we fit the cubic $f^j(t) = a^j_3 t^3 + a^j_2 t^2 + a^j_1 t + a^j_0$ through the points $(-3\delta t, Z^j_{-3}), (-2\delta t, Z^j_{-2}), (-\delta t, Z^j_{-1})$ and $(0, Z^j_0)$. The extrapolated value of Z^j is then $f^j(N_2 \delta t)$. The top panel of Fig. 3 shows the speedup as a function of N_2. The speedup is defined as the time taken to directly integrate (1)-(2) over $0 < t < 100$ with time-step δt divided by the time taken to integrate over $0 < t < 100$ using coarse projective integration, as described. A speedup greater than 1 (N_2 greater than approximately 10) means that

4 The 1:1 orbit

Consider the case of 1:1 locking, i.e. solutions for which each oscillator undergoes one oscillation during each forcing period. The usual way to study this would be to "strobe" the system once each forcing cycle. Defining X_p to be the state of the system at $t = 2\pi p/\omega$, where p is an integer, i.e.

$$X_p = X(2\pi p) \in R^{2N} \tag{11}$$

where X is defined in (9), we could construct a map $g : R^{2N} \rightarrow R^{2N}$ as

$$X_{p+1} = g(X_p) \tag{12}$$

A 1:1 locked orbit is then a fixed point of g and its stability is determined by the eigenvalues of the Jacobian of g

5.2 Varying β

lators (those with the highest values of μ) are not synchronised with the main cluster, either inside or outside of the tongue. However, the remaining □ 99% are synchronised with each other and using the "macroscopic" approach taken here we can detect whether this large cluster is synchronised with the forcing signal or not.

As β is increased, the fraction of oscillators no longer locked to the main cluster increases and the description of the system from the macroscopic point of view as a forced super-oscillator, using polynomial chaos coe□cients, becomes increasingly flawed. This is the reason for deciding to terminate the curves in Fig. 8. Note that the two curves in Fig. 8 terminate at di□erent values of β, but for both curves, the saddle-node bifurcation following algorithm fails to converge within tolerances when approximately 1% of the oscillators become desynchronised from the main group.

saddle-node bifurcation of periodic orbits. This walkthrough occurs approximately periodically, and the period scales as $|\omega - \omega^*|^{-1/2}$, where ω^* is the value of ω at the relevant tongue boundary [11]. As can be seen, this slow oscillation can be made arbitrarily slow by adjusting ω.

A similar phenomenon occurs in our system, but with a slight d

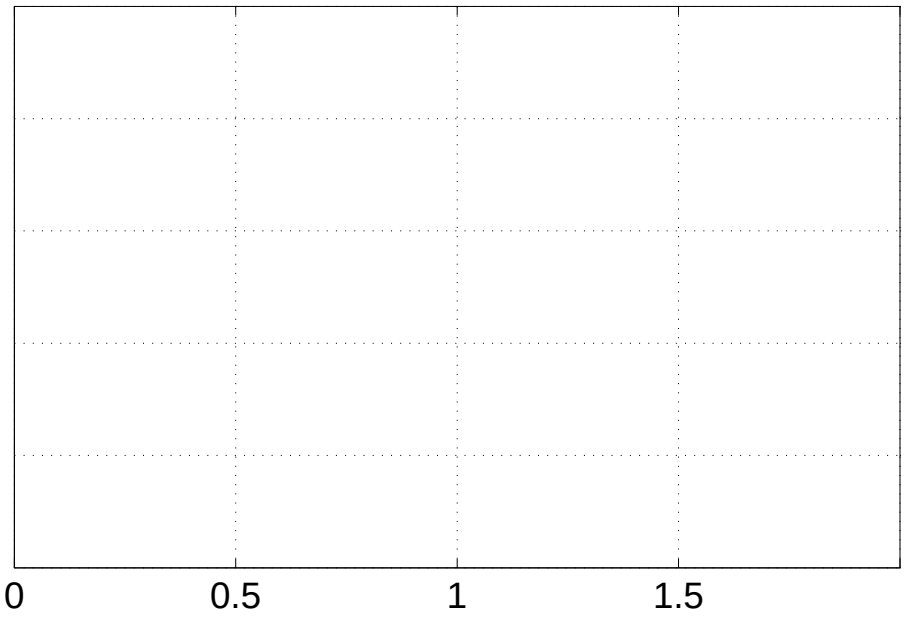
curves using standard algorithms [10], and the results for the left cusp are shown in Fig. 12, both for a single oscillator and for the inhomogeneous network of 500 oscillators. (We also followed the Hopf bifurcation curve associated with the right cusp, not shown.) The Hopf bifurcations correspond to a complex conjugate pair of eigenvalues crossing out of the unit circle in the complex plane as ω is decreased. Writing these eigenvalues at bifurcation as $e^{\pm i\theta}$, we have $\theta = 0$ at the rightmost point of the Hopf bifurcation curve (i.e. eigenvalues of +1, +1) and θ monotonically increases as ω is decreased until $\theta = \pi$ (i.e. eigenvalues of −1, −1) at the leftmost point on the Hopf bifurcation curve.

Note that while following the curve of Hopf bifurcations for the network, we could not use a larger value of β than $\beta \approx 0.3$ (i.e. we could not use a more heterogeneous network) because for larger values of β the oscillators with the largest values of μ would become desynchronised from the rest as the bifurcation was approached. The problem discussed in Sec. 5.2 regarding the e□ectiveness of the macroscopic approach would then reoccur.

The Hopf bifurcation for a single forced oscillator is super

The main result of this paper is the demonstration that the dynamics of large, heterogeneous networks of forced coupled oscillators can be simulated and analysed using a low-dimensional description of the state of the network, provided that all or most of the oscillators are synchronised with one another. The reduction to a low-dimensional description is performed "on the fly" and results in considerable computational simplification. For this specific network and heterogeneity we found that the e□ects of heterogeneity are to move the

References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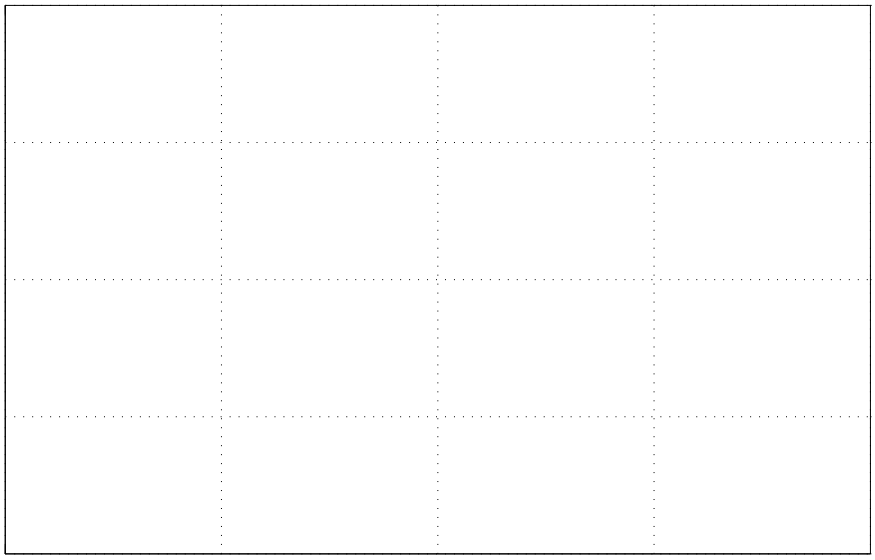

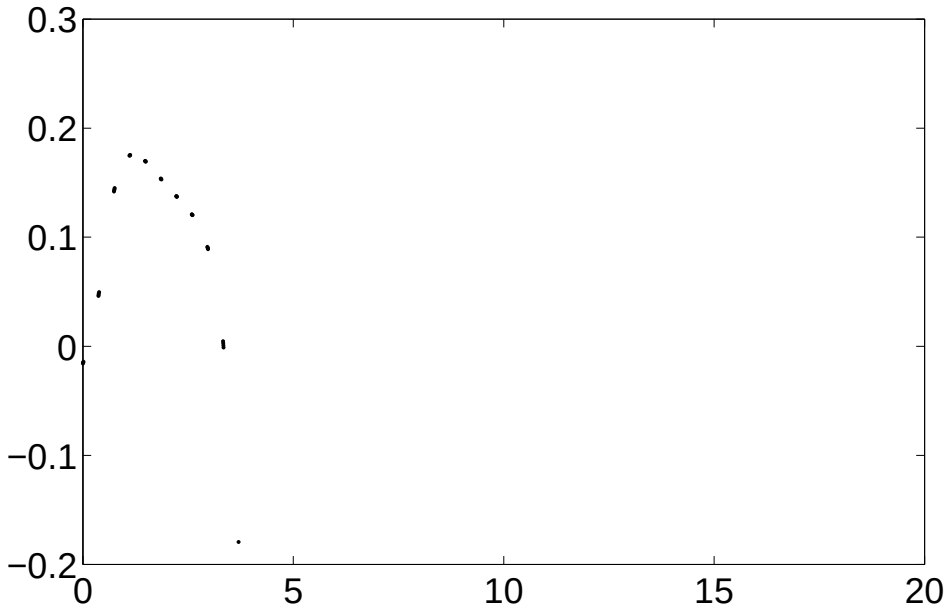
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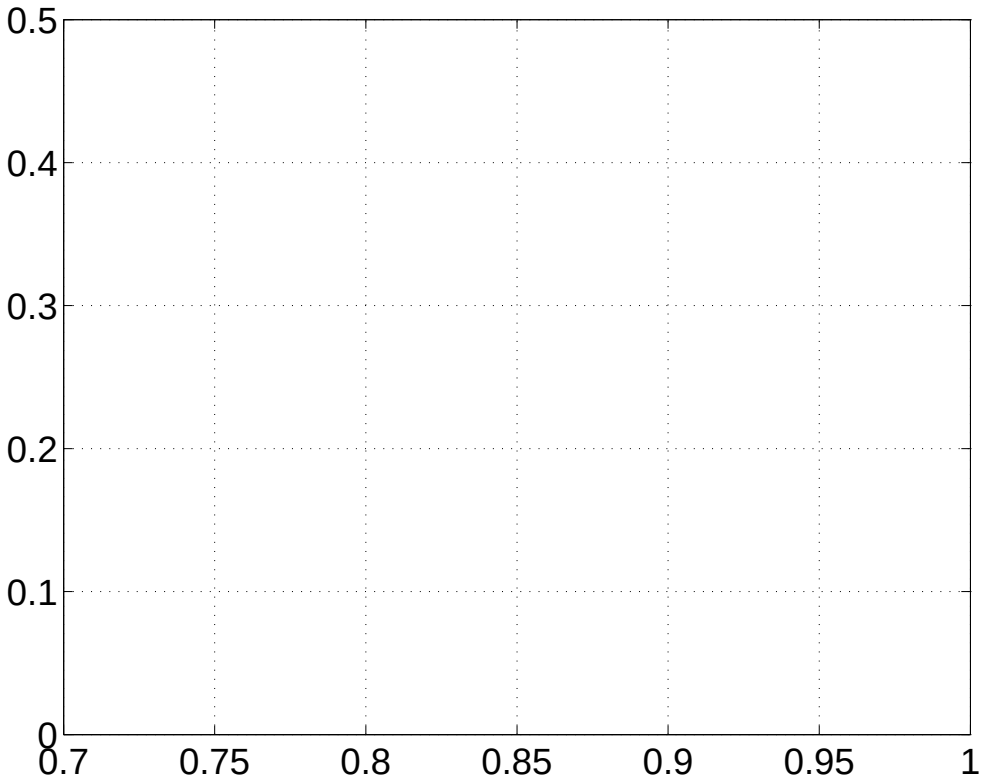
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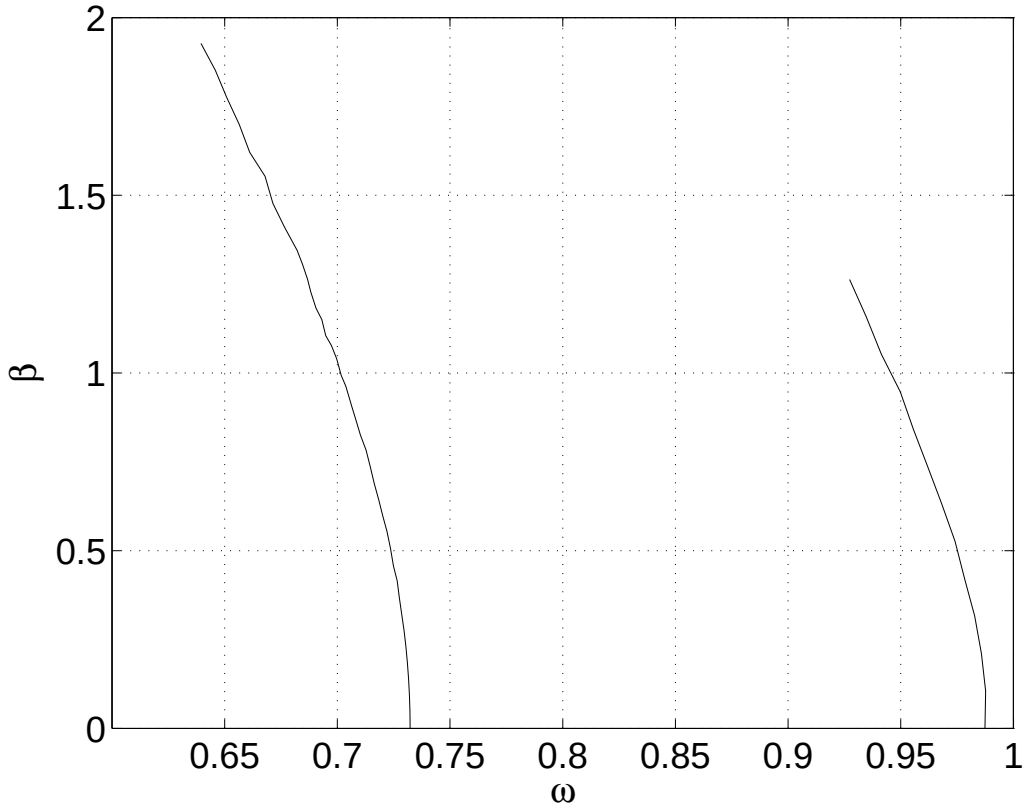
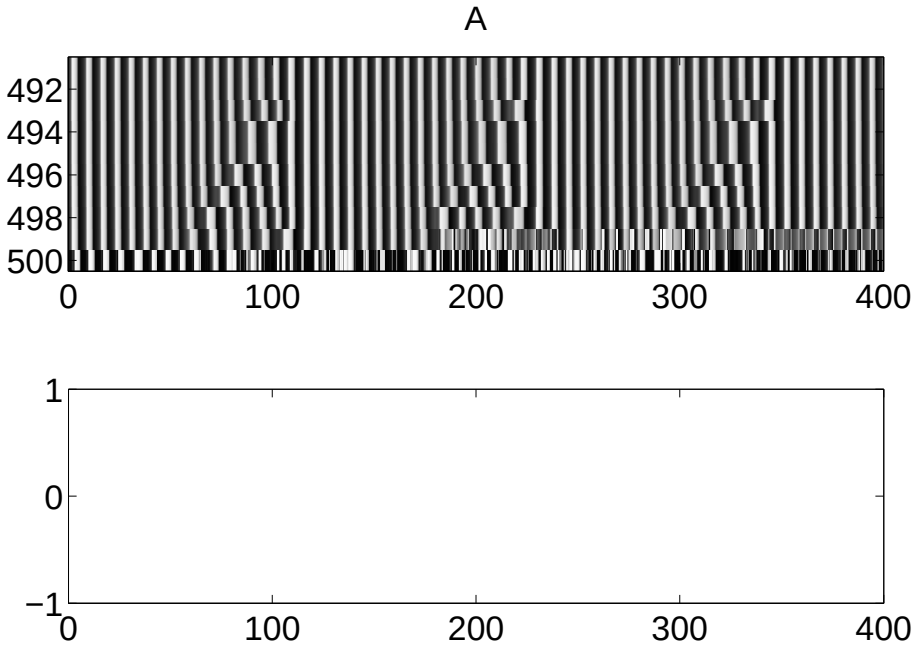


Fig. 8. Boundaries of the 1:1 Arnold tongue, averaging over between 20 and 60 realisations. The curves terminate as β is increased because the oscillators become too heterogeneous to synchronise among themselves. N = 500 and A = 0.5. Other parameters are = 1, φ = 1.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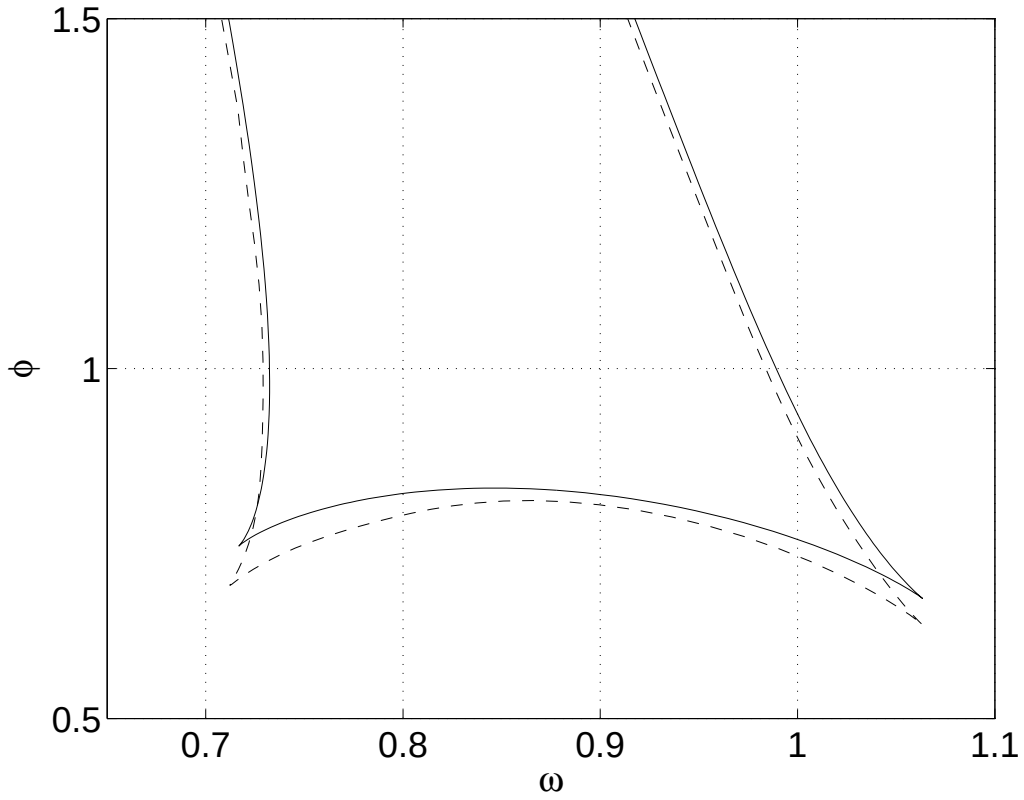


Fig. 10. Boundaries of the 1:1 orbit. Solid line: one oscillator. Dashed line: a network with N = 500 and A = 0.5, β = 0.3, r = 20, □ = 1.

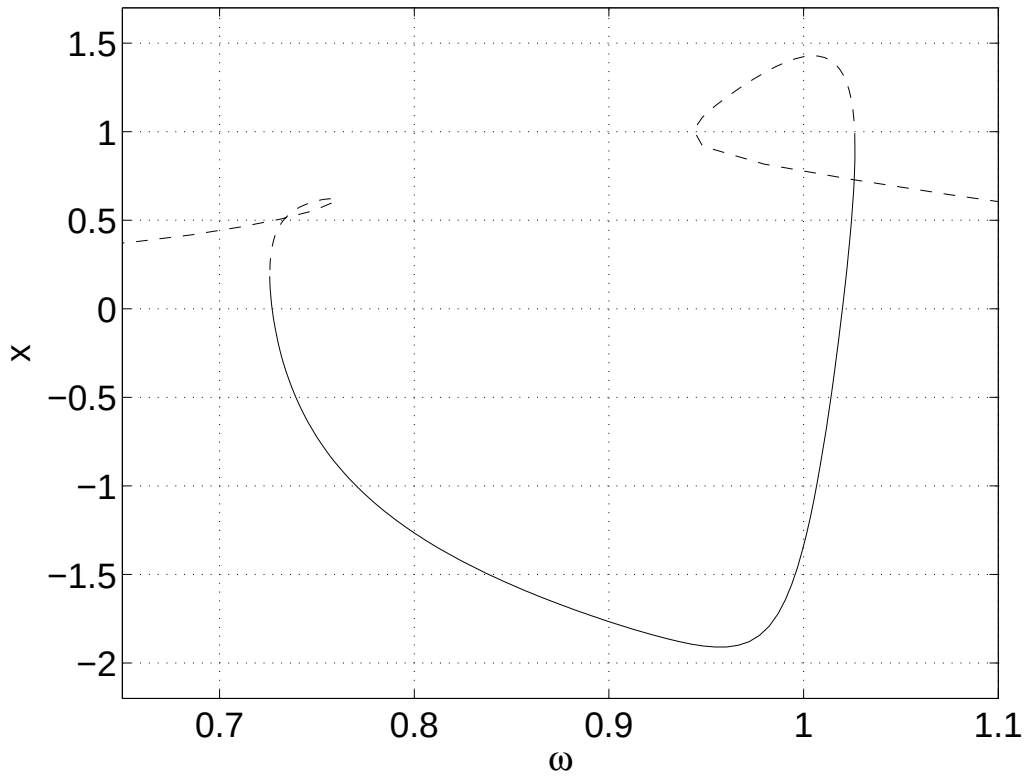
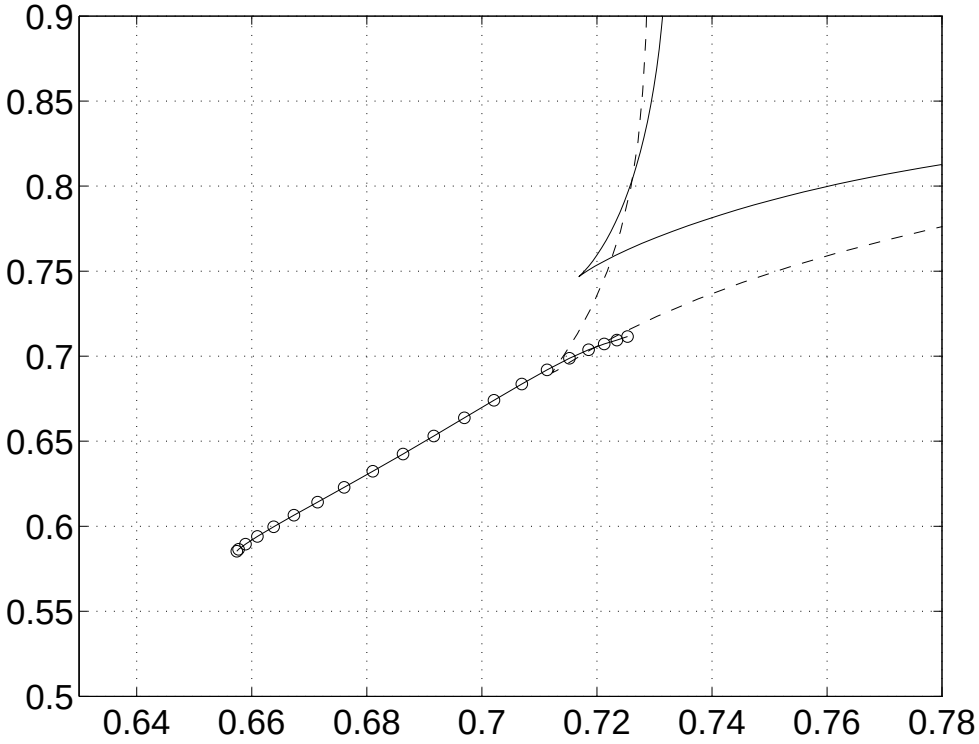


Fig. 11. x at multiples of $2\pi/\omega$ for the 1:1 orbit of a single oscillator, as ω is varied. Solid line: stable, dashed: unstable. Parameters are $A = 0.5, \varphi = 0.8$. This figure is a horizontal slice at $\varphi = 0.8$ through Fig. 10.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